PROBABILITY

THEORY & PROBLEMS

I. THEORY

1. **Random experiment**: is a process or activity which produces a number of possible outcomes. The outcomes which result cannot be predicted with absolute certainty.

2. **Sample space** Ω: is a list of all possible outcomes of the experiment. The outcomes must be mutually exclusive and exhaustive. Mutually exclusive means they are distinct and non-overlapping. Exhaustive means complete.

3. **Event** A: is a subset of the sample space. An event can be classified as a SIMPLE EVENT or COMPOUND EVENT.

   (a) Ω is referred to as THE SURE EVENT.

   (b) $\emptyset$ is referred to as the IMPOSSIBLE EVENT.

   (c) UNION OF EVENTS: For any two events A and B, the event $A \cup B$ consists of all outcomes that are either in A or in B, meaning that $A \cup B$ is realized if either A or B occurs.

   (d) INTERSECTION: For any two events A and B, the event $A \cap B$ consists of all outcomes that are both in A and in B, meaning that $A \cap B$ is realized if both A and B occur.

   (e) A and B are said to be MUTUALLY EXCLUSIVE if $A \cap B = \emptyset$: $A \cap B$ is the impossible event, meaning that A and B cannot both occur in the same time.

   (f) For any event $A \subset \Omega$, we define the event A' or (A^c), referred to as the COMPLEMENT OF A, to consist of all outcomes in the sample space Ω that are not in A, meaning that A' is realized if A does not occur. Note that $A \cap A' = \emptyset$ and $A \cup A' = \Omega$.

4. **The theoretical probability (also known as Classical approach) of an event** $A \subset \Omega$ is defined as the number of ways the event A can occur divided by the number of events of the sample space Ω. Using mathematical notation, we have

$$P(A) = \frac{|A|}{|\Omega|}$$

   where $|A|$ is the number of ways the event can occur and $|\Omega|$ represents the total number of events in the sample space.

5. AXIOMS & PROPERTIES OF PROBABILITY

   (a) **Probability Axioms**
   Assume that $A, B \subset \Omega$

   - (A1) The probability of is non-negative.

$$P(A) \geq 0.$$

   - (A2) The probability of the sure event is 1.

$$P(\Omega) = 1$$

   - (A3) If A and B are mutually exclusive ($A \cap B = \emptyset$), then:

$$P(A \cup B) = P(A) + P(B).$$

(b) **Probability Properties**

- (P1) The probability of an impossible event is zero.

$$P(\emptyset) = 0.$$

- (P2) If an event is a subset of another event, its probability is less than or equal to it.

$$A \subset B \Longrightarrow P(A) \leq P(B)$$

- (P3) The sum of the probabilities of an event and its complementary is 1, so the probability of the complementary event is:

$$P(A) + P(A') = 1$$

- (P4) The probability of the union of two events is the sum of their probabilities minus the probability of their intersection.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

- (P5) If $A_1, A_2, \ldots, A_k$ are mutually exclusive between them, then:

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots + P(A_k)$$

6. **Conditional Probability** The probability of event B GIVEN EVENT A (or THE PROBABILITY OF A UNDER THE CONDITION B) equals the probability of event A and event B divided by the probability of event A.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

7. **Total Probability Theorem** Given n mutually exclusive events $B_1, B_2 \ldots, B_n \subset \Omega$, such that

   - $B_1 \cup B_2 \cup \ldots \cup B_n = \Omega$ (a partition of the sample space Ω),
   - $P(B_1) + P(B_2) + \ldots + P(B_n) = 1$, (whose probabilities sum to unity)

   then

$$P(A) = P(A|B_1)P(B_1) + \ldots + P(A|B_n)P(B_n),$$

   where $A \subset \Omega$ is an arbitrary event, and $P(A|B_i)$ is the conditional probability of A assuming B_i, for $i = 1, 2, \ldots n$.

8. **Bayes Theorem** Let $B_1, \ldots, B_n$ be a partition of Ω. For any event A

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(B)}$$

II. PROBLEMS

1. A coin is tossed twice, then what will be the probability of getting one head and one tail?
2. A bag has 3 red, 2 green and 4 black ball. If a ball is randomly taken from the bag, find the theoretical probability that it is green.
3. Two number cubes are thrown. What is the theoretical probability of rolling a number greater than 8?

4. There are an equal number of pennies, nickels, dimes, and quarters in a bag. What is the probability that the combined value of the four coins randomly selected with replacement will be $0.41?

5. A party host gives a door prize to one guest chosen at random. There are 48 men and 42 women at the party. What is the probability that the prize goes to a woman?

6. What is the probability of getting a license plate that has a repeated letter or digit if you live in a state where the license plate scheme is four letters followed by two nume?

7. 70% of your friends like Chocolate, and 35% like Chocolate AND like Strawberry. What is the probability that randomly chosen friend who likes Chocolate also likes Strawberry?

8. You are off to soccer, and want to be the Goalkeeper, but that depends who is the Coach today:

   - with Coach Sam the probability of being Goalkeeper is 0,5
   - with Coach Alex the probability of being Goalkeeper is 0,3

   Sam is Coach more often ... about 6 out of every 10 games (a probability of 0,6). So, what is the probability you will be a Goalkeeper today?

9. In a class, 40% of the students study math and science. 60% of the students study math. What is the probability of a student studying science given he/she is already studying math?

10. At a certain university, 4% of men are over 6 feet tall and 1% of women are over 6 feet tall. The total student population is divided in the ratio 3:2 in favour of women. If a student is selected at random from among all those over six feet tall, what is the probability that the student is a woman?

11. A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from

    (a) machine A

    (b) machine B

    (c) machine C?

SOURCES:

1. https://www.mathsisfun.com/data/probability

2. https://www.mathsisfun.com/data/probability-events-conditional.html

3. http://www3.govst.edu/kriordan/files/ssc/math161/pdf/Chapter7ppt.pdf

4. '100 Probability Questions Prob-Stats (Math 3350)'

5. Neh emy Lim, 'STAT/MATH 394 A - PROBABILITY I UW', Autumn Quarter 2016