1.1 REAL NUMBERS
THEORY & EXERCISES

1. BASIC OPERATIONS

   ($+$) — **ADDITION — addend plus addend equals sum**
   ($-$) — **SUBTRACTION — minuend minus subtrahend equals difference**
   ($\times$) — **MULTIPLICATION — factor times factor equals product**
   ($\div$) — **DIVISION — dividend divided by divisor equals quotient**

   (a) THE BASIC ARITHMETIC PROPERTIES

      i. **Commutative Property** - The commutative property describes equations in which the order of the numbers involved does not affect the result. Addition and multiplication are commutative operations:

         $123 + 324 = 324 + 123$
         $24 \times 44 = 44 \times 24$

         Subtraction and division, however, <u>are not commutative</u>.

      ii. **Assosiative Property** - The associative property describes equations in which the grouping of the numbers involved does not affect the result. As with the commutative property, addition and multiplication are associative operations:

         $13 + (34 + 21) = (13 + 34) + 21$
         $(14 \times 4) \times 10 = 14 \times (4 \times 10)$

         Once again, subtraction and division are not <u>associative</u>.

      iii. **Distributive Property** - The distributive property can be used when the sum of two quantities is then multiplied by a third quantity.

         $(2 + 4) \times 3 = 2 \times 3 + 4 \times 3 = 18$

   (b) NEGATIVE NUMBERS - OPERATIONS ON NEGATIVE NUMBERS

      - The addition of two negative numbers results in a negative; the addition of a positive and negative number produces a number that has the same sign as the number of larger magnitude.
      - Subtraction of a positive number yields the same result as the addition of a negative number of equal magnitude, while subtracting a negative number yields the same result as adding a positive number.
      - The product of one positive number and one negative number is negative, and the product of two negative numbers is positive.
      - The quotient of one positive number and one negative number is negative, and the quotient of two negative numbers is positive.

2. THE SET OF NATURAL NUMBERS $\mathbb{N} = \{0, 1, 2, 3, ...\}$

(a) NATURAL NUMBERS

- **Prime numbers:** A number $n \in \mathbb{N}$, $n > 1$ is called prime if its only factors are 1 and itself.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$$

Theorem 1 (Euclid's theorem) *There are infinitely many prime numbers.*

PROOF:

Consider any finite list of prime numbers $p_1, p_2, \ldots, p_n$. It will be shown that at least one additional prime number not in this list exists. Let P be the product of all the prime numbers in the list: $P = p_1 \times p2 \times \ldots \times p_n$. Let $Q = P + 1$. Then Q is either prime or not. If Q is prime, then there is at least one more prime that is not in the list. If Q is not prime, then some prime factor p_i, $(i = 1, 2, \ldots, n)$ divides Q. If this factor p_i were in our list, then it would divide P (since P is the product of every number in the list); but P divides $P + 1 = Q$. If p_i divides P and Q, then p_i would have to divide the difference of the two numbers, which is $(P + 1)P$ or just 1. Since no prime number divides 1, p_i cannot be on the list. This means that at least one more prime number exists beyond those in the list. This proves that for every finite list of prime numbers there is a prime number not in the list, and therefore there must be infinitely many prime numbers. *q.e.d.*

- **Composite numbers:** A number is called composite if it is not prime, that is if it has more than two distinct factors.

Theorem 2 (Unique Factorization Theorem) *Every natural number greater than 1 is a power of a prime or can be expressed as a product of powers of primes. This factorization is unique, apart from the ordering. That is, If we ignore the order in which we write the prime factors there is only one prime factorization of every natural number.*

- **GCD(a,b)** - The Greatest Common Devisor
- **LCM(a,b)** - The Lowest Common Multiple

(b) DIVISIBILITY OF NATURAL NUMBERS

- **Even numbers** – divisible by 2 – $0, 2, 4, 6, 8, 10, 12, \ldots$
- **Odd numbers** – not divisible by 2 – $1, 3, 5, 7, 9, 11, \ldots$

3. THE SET OF WHOLE NUMBERS (INTEGERS) $\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

Theorem 3 *For any whole numbers a and b we know (by say, the repeated subtractionor division algorithm) that there are unique whole numbers q and r so that $a = b \times q + r$, where $0 \leq r < b$.*

Here q is called **the quotient** and r is called **the remainder**.

(a) DIVISIBILITY RULES

- `Divisibility by 0` — No numbers are divisible by 0

- `Divisibility by 1` — All numbers are divisible by 1.

- `Divisibility by 2` — Even numbers are divisible by 2.
  Example: 109850 is divisible by 2 because it is an even number.

- `Divisibility by 3` — Add the digits of a number together. If the sum is divisible by 3, then the original number is divisible by 3.
  Example: The number 792 is divisible by 3 because $7+9+2 = 18$, and 18 is divisible by 3.

- Divisibility by 4 — If the last two digits of a number are divisible by 4, then the original number is divisible by 4.
  Example: The number 16248 is divisible by 4 because the last two digits, 48, are divisible by 4.
- `Divisibility by 5` — If a number ends in 0 or 5, then the number is divisible by 5.
  Example:The number 563,021,689,540 is divisible by 5 because it ends in 0.

- `Divisibility by 6` — If a number is divisible by 2 and 3, then it is also divisible by 6.
  Example: The number 6874 is not divisible by 6, even though 6874 is even, indicating divisibility by 2, but $6 + 8 + 7 + 4 = 25$, and 25 is not divisible by 3.

- `Divisibility by 7` — Double the last digit and then subtract it from the number formed by the remaining digits.If the result is divisible by 7 or equal to 0, then the original number is divisible by 7. This can be repeated if necessary.
  Example: The number 3416 is divisible by 7 because: Double the last digit $6 \times 2 = 12$. Subtract the result from remaining digits $341 - 12 = 329$ Repeat if necessary with the result. In this case 329. $9 \times 2 = 18$, then $32 - 18 = 14$, and 14 is divisible by 7.

- `Divisibility by 8` — If the last three digits of a number are divisible by 8, then the original number is divisible by 8.
  Example: The number 5128 is divisible by 8 because $128 \div 8 = 16$, and 16 is divisible by 8.

- `Divisibility by 9` — Add the digits of a number together. If the sum is divisible by 9, then the original number is divisible by 9.

Example: The number 65762 is not divisible by 9 because $6+5+7+6+2=26$, and 26 is not divisible by 9.

- `Divisibility by 10` — If the number ends in 0, then it is divisible by 10.
  Example: The number 29581940 is divisible by 10 because the last digit is a 0.

- `Divisibility by 11` — Alternately add and subtract the digits of the number. If the result is divisible by 11 or equal to 0 then the original number is divisible by 11.
  Example: The number 3564 is divisible by 11 because $3-5+6-4=0$.

- `Divisibility by 12` — If a number is divisible by 3 and 4, then it is also divisible by 12.
  Example: The number 409536 is divisible by 12 because $4+0+9+5+3+6=27$ which shows divisibility by 3, and the last two digits, 36, indicate divisibility 4.

4. The SET OF RATIONAL NUMBERS (QUOTIENTS) $\mathbb{Q}=\{\frac{p}{q}, p\in\mathbb{Z}, q\in\mathbb{Z}-\{0\}\}$

   A rational number is a number that can be written in the form of **a fraction** $\frac{p}{q}$ such that p and q are integers and $q\neq 0$.

   (a) Decimal Expansion of the rational number
   Every rational number has a decimal expansion.
   - **terminating decimal expansion**: $\frac{7}{8}=0.875$
   - **repeating decimal expansion**: $\frac{4}{21}=0.(190476)=0.\overline{190476}$

5. The SET OF IRRATIONAL NUMBERS $I\mathbb{Q}$

   The set of irrational numbers is equal to the set of real numbers that have decimal expansion in the form of **non-terminating non-repeating decimals**.

   - $\sqrt{2}=1.41421356273095\ldots$
   - $\pi=3.14159265358979\ldots$
   - $e=2.718281828459045\ldots$

6. POWERS AND ROOTS

   (a) Indices

   - **Indices** (or **powers**, or **exponents**) are very useful in mathematics. Indices are a convenient way of writing multiplications that have many repeated terms.
     a^n - a to the power n
     a - the base
     n - the index, the exponent, the power (n may be any real number)
   - **Standard Form** $m=a\times 10^m$, $a\in[1;10)$, $m\in\mathbb{Q}$

   (b) Roots
   - **Roots** are the inverse operation of exponentiation. This means that if $\sqrt[n]{x}=r$, then $r^n=x$.

- **The square root** of a value is the number that when squared results in the initial value. In other words, $\sqrt{y} = x$ if $x^2 = y$.
- **The cube root** of a value is the number that when cubed results in the initial value. In other words, $\sqrt[3]{y} = x$ if $x^3 = y$.
- **Radical expression:** A mathematical expression that contains a root, written in the form $\sqrt[n]{a}$ – **a root of degree n of a.**

7. PROBLEMS

Problem 1. Find the GCD and LCM of numbers 1460 and 2048.
Problem 2. Find the remainder of division 2376 by 35.
Problem 3. Convert the fraction $\frac{7}{80}$ to a terminating decimal.
Problem 4. Convert the fraction $\frac{4}{33}$ to a recurring decimal.
Problem 5. Put the following decimals in order from lowest to highest: 0.23(4), 0.2(34), 0.(234).
Problem 6.Convert the recurring decimal 0.13(14) to a fraction.
Problem 7. Find a 148^{th} digit after the decimal point in the recurring decimal 0.12(3456789).