DIFFERENTIAL CALCULUS (1)

LIMITS & CONTINUOUS FUNCTIONS

VOCABULARY, EXERCISES & PROBLEMS

1. VOCABULARY

(a) **neighbourhood** On the real line, a neighbourhood of the real number a is an open interval $(a-\delta, a+\delta)$, where $\delta > 0$, with its centre at a.

(b) **limit** (of $f(x)$) The limit, if it exists, of $f(x)$ as x tends to x_0 is a number g with the property that, as x gets closer to x_0, $f(x)$ gets closer and closer to g. This is written

$$\lim_{x \to x_0} f(x) = g.$$

(c) **limit as x tends to x_0 from the left (right)** The statement that $f(x)$ tends to g as x tends to x_0 from the left (right) can be written:

$$\lim_{x \to x_0^-} f(x) = g \qquad (\lim_{x \to x_0^+} f(x) = g),$$

(d) **infinite limits** For a function whose values grow without bound, the function diverges and the usual limit does not exist. However, in this case one may introduce limits with infinite values. For example, the statement the limit of f, as x approaches x_0, is infinity, is denoted as:

$$\lim_{x \to x_0} f(x) = +\infty, \qquad \lim_{x \to x_0} f(x) = -\infty$$

(e) **asymptote** A line l is an asypmptote to a curve if the distance from a point P to the line l tends to zero as P tends to infinity along some unbounded part of the curve.

(f) **limits at infinity** Let f be a function defined on (a, ∞). Then

$$\lim_{x \to \infty} f(x) = g$$

means that the values of $f(x)$ can be made arbitrarily close to g by taking x sufficiently large.

(g) **the dividing out technique** should be applied only when direct substitution produces 0 in both the numerator and the denominator. An expression such as $\frac{0}{0}$ has no meaning as a real number. It is called an **indeterminate form** because you cannot, from the form alone, determine the limit. When you try to evaluate a limit of a rational function by direct substitution and encounter this form, you can conclude that the numerator and denominator must have a common factor. After factoring and dividing out, you should try direct substitution again.

(h) **rationalizing technique** Rationalizing the numerator means multiplying the numerator and denominator by the conjugate of the numerator. For instance, the conjugate of $\sqrt{x}+4$ is $\sqrt{x}-4$.

(i) **continuous function** The real function f of one variable is continuous at x_0 if $f(x)$ tends to $f(x_0)$ as x tends to x_0. A function f is continuous in an open interval if it is continuous at each point of te interval; and f is continuous on the closed interval $[a, b]$, where $a < b$, if it is continuous in the open interval (a, b) and if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b)$$

2. EXERCISES & PROBLEMS

(a) Write the properties of limits.

(b) Give a definition of the **vertical asymptote** and the **horizontal asymptote**.

(c) Evaluate the following limits:

i.
$$\lim_{x\to -1} \frac{x^2-1}{x+1}$$

ii.
$$\lim_{x\to 0} \frac{x^2-x}{3x}$$

iii.
$$\lim_{x\to 1} \frac{1-\sqrt{x+1}}{x}$$

iv.
$$\lim_{x\to 0} \frac{x+\sqrt{x}}{\sqrt{x}}$$

v.
$$\lim_{x\to 1} \frac{x^2+x-2}{x-1}$$

vi.
$$\lim_{x\to -1} \frac{x^3+x^2-x-1}{3x+3}$$

vii.
$$\lim_{x\to 0} \frac{x^2-x}{3x}$$

(d) Find the asymptotes (vertical and horizontal) of the following curves:

i.
$$f(x) = \frac{x^2}{x-2}$$

ii.
$$f(x) = \frac{x^2-x-1}{2x}$$

iii.
$$f(x) = \frac{x^3}{2(x+1)^2}$$

iv.
$$f(x) = \frac{2x^3-x^2+2x+2}{x^2+1}$$

(e) The function f is given by the formula

$$f(x) = \begin{cases} x-1, & \text{gdy } x \le 0 \\ 0, & \text{gdy } 0 < x < 2 \\ 2x-4, & \text{gdy } x \ge 2 \end{cases}$$

Sketch the graph of the function. Is the function f continuous?

(f) The function $f(x) = \frac{x^2-1}{x-1}$ is not defined at $x = 1$. What value is given to the function f at $x = 1$, knowing that the function f is continuous at $x = 1$?

(g) Proof that the function

$$f(x) = \begin{cases} \frac{x}{|x|} & \text{gdy } x \neq 0 \\ 0 & \text{gdy } x = 0 \end{cases}$$

is discontinuous.

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