4. Sequences

Task 4.01. (0-2) (2015 – task 12)

The sequence (a_n) is an arithmetic sequence defined for $n \geq 1$, with $a_1 = -3$ and $a_5 = 9$. Complete the following sentences.

(a) The tenth term of the arithmetic sequence is equal to ....

(b) The sum of the first ten terms of the arithmetic sequence is equal to ....

Solution 4.01. (a) 24 (b) 105

$a_5 = a_1 + 4r$

$9 = -3 + 4r$

$4r = 12$

$r = 3$

(a) $a_{10} = a_1 + 9r = -3 + 27 = 24$

(b) $S_{10} = \frac{a_1 + a_{10}}{2} \times 10 = \frac{-3+24}{2} \times 10 = 105$

Task 4.02. (0-2) (20161– task 16)

The seventeenth term of a geometric sequence equals 10, while its twentieth term equals −80. Complete the following sentences.

a) The common ratio of this geometric sequence is …………….. .

b) The number of terms in this sequence which are in the interval $(0, 1)$ equals …………….. .

Solution 4.02. (a) -2 (b) 7

Let (a_n) be the given geometric sequence and let q be its common ratio.

$$\begin{cases} a_{17} = 10 \\ a_{20} = -80 \end{cases} \quad \begin{cases} a_1 \times q^{16} = 10 \\ a_1 \times q^{19} = -80 \end{cases} \quad \begin{cases} a_1 = \frac{10}{q^{16}} \\ \frac{10}{q^{16}} \times q^{19} = -80 \end{cases} \quad \begin{cases} a_1 = \frac{10}{q^{16}} \\ q^3 = -8 \end{cases}$$

$$\begin{cases} a_1 = \frac{10}{(-2)^{16}} \\ q = -2 \end{cases} \quad \begin{cases} a_1 = \frac{10}{2^{16}} \\ q = -2 \end{cases} \quad \begin{cases} a_1 = \frac{5}{2^{15}} \\ q = -2 \end{cases}$$

(a) So now we know that the common q ratio is -2.

(b) Let's find the closed formula for the term a_n.

$a_n = a_1 \times q^{n-1}$

4. Sequences

$$a_n = \frac{5}{2^{15}} \times (-2)^{n-1}$$

$$a_n = \frac{5}{2^{15}} \times 2^{n-1} \times (-1)^{n-1}$$

$$a_n = (-1)^{n-1} \times 5 \times 2^{n-16}$$

Notice that:

- if n is an even number then $a_n = -5 \times 2^{n-16} < 0$.
- if n is an odd number then $a_n = 5 \times 2^{n-16} > 0$.

If $a_n \in (0,1)$ then n must be an odd number and $5 \times 2^{n-16} < 1$, which means that $2^{n-16} < \frac{1}{5}$.

Notice that $2^{-3} < \frac{1}{5} < 2^{-2}$, so the inequality will be satisfied if $2^{n-16} \leq 2^{-3}$

Therefore $n - 16 \leq -3$ and so $n \leq 13$.

So $a_n \in (0,1)$ if and only if n is an odd number and $n \leq 13$, which means that only 7 terms satisfy this condition: $a_1, a_3, a_5, a_7, a_9, a_{11}, a_{13}$.

Task 4.03. (0-1) (2017 – task 07)

Numbers $2x, 4x, 18$ (in the given order) are the first three terms of an arithmetic sequence. The first term of the sequence is:

A. 2.25 **B.** 1.5 **C.** 6 D. 3

Solution 4.03. D

$4x - 2x = 18 - 4x$

$6x = 18$

$x = 3$

Task 4.04. (0-3) (2017 – task 12)

4. Sequences

The odd-numbered terms of a given geometric sequence (a_n), where $n \geq 1$, are negative numbers and $a_5 = -1$. Even-numbered terms of the sequence are positive numbers and $a_{10} = \frac{1}{32}$. Complete the following sentences.

(a) The common ratio q of the geometric sequence (a_n) is ....

(b) In the geometric sequence (a_n), the number of terms greater than $\frac{1}{32}$ is ....

(c) In the sequence (a_n), the sum of integer terms is ....

Solution 4.04. (a) $-\frac{1}{2}$ (b) **4** (c) $\mathbf{-11}$

Let q be its common ratio of the sequence (a_n) .

$a_1 < 0$ and $a_2 = a_1 \times q < 0$, so $q < 0$.

- Let's find out the first term a_1 and the common ratio q:

$$\begin{cases} a_5 = -1 \\ a_{10} = \frac{1}{32} \end{cases} \quad \begin{cases} a_1 \times q^4 = -1 \\ a_1 \times q^9 = \frac{1}{32} \end{cases} \quad \begin{cases} a_1 = \frac{-1}{q^4} \\ \frac{-1}{q^4} \times q^9 = \frac{1}{32} \end{cases} \quad \begin{cases} a_1 = \frac{-1}{q^4} \\ q^5 = -\frac{1}{32} \end{cases}$$

$$\begin{cases} a_1 = -16 \\ q = -\frac{1}{2} \end{cases}$$

- Let's find out the closed formula for the n-th term of the sequence

$$a_n = -16 \times \left(-\frac{1}{2}\right)^{n-1}$$

$$a_n = (-1) \times 2^4 \times (-1)^{n-1} \left(\frac{1}{2}\right)^{n-1}$$

$$a_n = (-1)^n \times \left(\frac{1}{2}\right)^{-4} \times \left(\frac{1}{2}\right)^{n-1}$$

$$a_n = (-1)^n \times \left(\frac{1}{2}\right)^{n-5}$$

- $a_n = (-1)^n \times 2^{5-n}$

  Therefore

  if n is an odd number then $a_n = -2^{5-n} < 0$,

  if n is an even number then $a_n = 2^{5-n} > 0$

- Now, if $a_n > \frac{1}{32}$ then n must be an even number and $2^{5-n} > \frac{1}{32}$

  $2^{5-n} > 2^{-5}$

  $5 - n > -5$

$n < 10$

$n \in \{2, 4, 6, 8\}$

Now we deduce that only 4 terms in the sequence are greater that $\frac{1}{32}$.

These are a_2, a_4, a_6, a_8 .

- $a_n = (-1)^n \times 2^{5-n}$ is an integer if and only if $n \leq 5$.

  So we can calculate the sum of all integer terms straightforward:

  $S_5 = a_1 + a_2 + a_3 + a_4 + a_5 = -16 + 8 - 4 + 2 - 1 = -11$

Task 4.05. (0-3)   (2018 – task 13)

The sequence (a_n) is a geometric sequence defined for $n \geq 1$, with $a_1 = \frac{1}{4}$ and $a_4 = 2$. Complete the following sentences.

a) The seventh term of the sequence is ....

b) The product of the second and the eighth term of the sequence is ....

c) If the sum of n initial terms of the sequence is equal to $\frac{7}{4}$, then the number n is equal to ....

Solution 4.04. (a) **16** (b) **16** (c) **3**

Let q be its common ratio of the sequence (a_n) .

$q^3 = \frac{a_4}{a_1} = \frac{2}{\frac{1}{4}} = 8$

$q = 2$

$a_n = a_1 q^{n-1} = \frac{1}{4} \times 2^{n-1} = 2^{n-3}$

(a) $a_7 = 2^{7-3} = 2^4 = 16$

(b) $a_2 \times a_8 = 2^{2-3} \times 2^{8-3} = 2^{-1} \times 2^5 = 2^4 = 16$

(c) $S_n = a_1 \times \frac{1-q^n}{1-q} = \frac{1}{4} \times \frac{1-2^n}{1-2} = \frac{2^n-1}{4}$

$\frac{2^n-1}{4} = \frac{7}{4}$

$2^n = 8$

$n = 3$

Task 4.06. (0-3) (2018 – task 17)

In an arithmetic sequence (a_n) defined for all natural numbers such that $n \geq 1$, the first term is $a_1 = -7$ and the sum of the first twenty terms equals $S_{20} = 1000$. Complete the following sentences.

a) The common difference of this arithmetic sequence is …………………… .

b) The twentieth term of this sequence is …………………… .

c) The n −th term of this sequence is given by the formula: $a_n =$ …………………… .

Solution 4.06. (a) $\mathbf{6}$ (b) $\mathbf{107}$ (c) $\mathbf{6n - 13}$

Let r be its common difference of the sequence (a_n) .

$S_{20} = \frac{a_1 + a_{20}}{2} \times 20$

$\frac{-7 + a_{20}}{2} \times 20 = 1000$

$-7 + a_{20} = 100$

$a_{20} = 107$

$a_1 + 19r = 107$

$-7 + 19r = 107$

$19r = 114$

$r = 6$

$a_n = a_1 + (n-1)r = -7 + 6(n-1) = 6n - 13$

Task 4.07. (0-3) (2019 – task 06)

In a decreasing geometric sequence (a_n) defined for each natural number $n \geq 1$, the ninth term equals 9, and the seventh term equals 81. Therefore the common ratio q of this sequence

A. $-\frac{1}{3}$ **B.** $\frac{1}{3}$ **C.** 3 **D.** -3

Solution 4.07. C

$q^{9-7} = \frac{a_9}{a_7}$

$q^2 = \frac{81}{9}$

$q^2 = 9$

The sequence is monotonic (because it's decreasing), so the common ratio must be a positive number, and thus $q = 3$.

Task 4.08. (0-1) (2019 – task 12

The n–th term of the sequence (a_n) is given by $a_n = \frac{7-2n}{3}$

for each natural number $n \geq 1$. Therefore this sequence is:

A. an arithmetic sequence, and its common difference equals $r = -\frac{2}{3}$.

B. an arithmetic sequence, and its common difference equals $r = \frac{7}{3}$.

C. a geometric sequence, and its common ratio equals $q = -\frac{2}{3}$.

D. a geometric sequence, and its common ratio equals $q = \frac{7}{3}$.

Solution 4.08. A

$a_n = \frac{7-2n}{3} = -\frac{2}{3}n + \frac{7}{3}$

$a_{n+1} = -\frac{2}{3}(n+1) + \frac{7}{3} = -\frac{2}{3}n + \frac{5}{3}$

$a_{n+1} - a_n = -\frac{2}{3}$

Task 4.09. (0-3) (2019 – task 16)

The fortieth term of an arithmetic sequence is 40. The sum of the first forty terms of this sequence also equals 40.

Complete the following sentences with the correct numbers.

a) The first term of the sequence is …………… .
b) The common difference of this arithmetic sequence is …………………… .
c) The number of negative terms in the sequence is …………… .

Solution 4.09. (a) $\mathbf{-38}$ (b) **2** (c) **19**

Let r be its common difference of the sequence (a_n) .

$a_{40} = 40$

$S_{40} = \frac{a_1 + a_{40}}{2} \times 40$

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$\frac{a_1+40}{2} \times 40 = 40$

$a_1 + 40 = 2$

$a_1 = -38$

$a_1 + 39r = 40$

$-38 + 39r = 40$

$r = 2$

$a_n = a_1 + (n-1)r = -38 + 2(n-1) = 2n - 40$

$2n - 40 < 0$

$n < 20$

So there are 19 negative terms.

Task 4.10. (0-1)   (2020 – task 12)

The sequence (a_n) is given by the formula $a_n = -n^2 + 14n - 42$ for $n \geq 1$. The number of its positive terms is:

A. 0   **B.** 3   **C.** 5   D. 12

Solution 4.10. C

$-n^2 + 14n - 42 > 0$

$n^2 - 14n + 42 < 0$

$\Delta = 196 - 168 = 28$

$\sqrt{\Delta} = \sqrt{28} = 2\sqrt{7}$

$n_1 = \frac{14-2\sqrt{7}}{2} = 7 - \sqrt{7} \approx 4.4$ $n_2 = \frac{14+2\sqrt{7}}{2} = 7 + \sqrt{7} \approx 9.6$

$n \in \{5, 6, 7, 8, 9\}$

So there are 5 positive terms in the sequence.

Task 4.11. (0-1)   (2020 – task 13)

In a geometric sequence (a_n) defined for $n \geq 1,\ a_2 = 1,\ a_3 = 1 + \sqrt{5}$. Therefore a_1 is equal to:

A. $\sqrt{5} - 1$ **B.** $\frac{\sqrt{5}-1}{4}$ **C.** $\sqrt{5} + 1$ **D.** $\frac{\sqrt{5}+1}{4}$

Solution 4.11. B

$\frac{a_2}{a_1} = \frac{a_3}{a_2}$

$\frac{1}{a_1} = \frac{1+\sqrt{5}}{1}$

$a_1 = \frac{1}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{\sqrt{5}-1}{(\sqrt{5})^2-1} = \frac{\sqrt{5}-1}{4}$

Task 4.12 (0-1) (2021 – task 09)

The fourth term of an arithmetic sequence is 7, and the seventh term is 4. The common difference of this arithmetic sequence is equal to

A. -3 **B.** -1 **C.** 1 D. 3

Solution 4.12. B

Let r be the common difference of the arithmetic sequence a_n.

$a_7 - a_4 = 3r$

$4 - 7 = 3r$

$r = -1$

Task 4.13 (0-4) (2021 – task 16)

A sequence (a_n) is given by the formula $a_n = 3n - 5$ for $n \geq 1$. Let T be a set of all two-digit numbers which are terms of the sequence (a_n).

Complete the sentences a – d below by writing the correct numeric values in the blanks.

a) The set T has .... elements.

b) The arithmetic mean of the elements of the set T is equal to ....

c) The median of the elements of the set T is equal to ....

d) The set T contains .... numbers which are squares of integers.

Solution 4.13. (a) 30 (b) 53.5 (c) 53.5 (d) 4

$10 \leq 3n - 5 \leq 99$

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$15 \leq 3n \leq 104$

$5 \leq n \leq 34$

$T = \{a_5, a_6, \dots, a_{34}\}$

$|T| = 34 - (5-1) = 30$

$S_T = \frac{a_5 + a_{34}}{2} \times 30 = \frac{10+97}{2} \times 30 = 53.5 \times 30$

Arithmetic mean = $\frac{S_T}{30} = \frac{53.5 \times 30}{30} = 53.5$

Median is the arithmetic mean of two central data $m = \frac{a_{19}+a_{20}}{2} = \frac{52+55}{2} = 53.5$

$a_n = 3n - 5 = c^2$ so $c^2 + 5$ must be divisible by 3.

$4^2 + 5 = 21 = 3 \times 7$

$5^2 + 5 = 30 = 3 \times 10$

$6^2 + 5 = 41$ is not divisible by 3

$7^2 + 5 = 54 = 3 \times 18$

$8^2 + 5 = 69 = 3 \times 23$

$9^2 + 5 = 86$ is not divisible by 3

As we can see there are 4 numbers in the set T that are perfect squares:

$a_7 = 16$, $a_{10} = 25$, $a_{18} = 49$, $a_{23} = 64$

Answers

4.01. (a) 24 (b) 105

4.02. (a) -2 (b) 7

4.03. D

4.04. (a) $-\frac{1}{2}$ (b) 4 (c) -11

4.05. (a) 16 (b) 16 (c) 3

4.06. (a) 6 (b) 107 (c) $6n - 13$

4.07. C

4.08. A

4.09. (a) -38 (b) 2 (c) 19

4.10. C

4.11. B

4.12. B

4.13. (a) 30 (b) 53.5 (c) 53.5 (d) 4